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ON THE CONVERGENCE OF (0,1,2) INTERPOLATION

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Abstract. For the Hermite interpolation polynomial, $H_m(x)$ we prove for any function $f \in C^{(2q)}([-1,1])$ and any $s = 0, 1, 2, \ldots, q$, where q is a fixed integer that

$$|H_m^{(s)}(x) - f^{(s)}(x)| = O(1)\omega(\frac{1}{m}, f^{(2q)})\frac{\log n}{n^{2q-2s}}.$$

Here m is defined by $m = 3n - 1$.

If $f \in C^{(q)}([-1,1])$, then

$$|H_m^{(s)} - f^{(s)}(x)| = O(1)\omega(\frac{1}{m}, f^{(q)})\frac{\log n}{(1-x^2)^{q/2}}$$

for $x \in (-1,1)$.

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1. Introduction. Suppose we have the triangular matrix

$$A(T) : \{x_{i,n}\}_{i=1}^{n}, n = 1, 2, 3, \dots . \tag{1.1}$$

where

$$x_{i,n} = \cos\frac{2i-1}{2n}\pi, \qquad 1 \le i \le n;\ n = 1, 2, 3, \dots , \tag{1.2}$$

are the roots of Tchebysheff polynomial

$$T_n(x) = \cos(n \arccos x), \qquad n = 1, 2, 3, \dots . \tag{1.3}$$

Corresponding to the matrix (1.1), suppose we have the matrices

$$M = \{m_{i,n}\}_{i=1}^{n} \tag{1.4}$$

where $m_{i,n} = 3$ and

$$Y = \left\{f_{i,n}^{(s)}\right\}_{i=1}^{n}, \qquad s = 0, 1, 2. \tag{1.5}$$

Here $f(x)$ is a real function defined on $[-1, 1]$ and $f_{i,n}^{(s)} = f^{(s)}(x_{i,n})$.

From theory of interpolation [4], we know that for given function $f(x)$ there exists an interpolation polynomial $H_{3n-1}(x, Y, A)$ of explicit form such that

$$H_{3n-1}^{(s)}(x_i, Y, A) = f_i^{(s)} \tag{1.6}$$

for $s = 0, 1, 2$ and $i = 0, 1, 2, \dots n$. In the last equation we have dropped the second index n and will be dropped in further equations. The explicit form of $H_{3n-1}(x, Y, A)$ is given by the following formula

$$H_{2n-1}(x, Y, A) = \sum_{i=1}^{n} f_i r_i(x) + \sum_{i=1}^{n} f_i' q_i(x) + \sum_{i=1}^{n} f_i'' z_i(x) \tag{1.7),}$$

where

$$r_i(x) = \left\{1 - \frac{3}{2}(x - x_i) - \frac{3}{2}\left[\frac{x_i^2}{(1-x_i^2)^2} + \frac{n^2-1}{2(1-x_i^2)}\right](x - x_i)^2\right\} l_i^3(x), \tag{1.8}$$

$$q_i(x) = \left\{(x - x_i) - \frac{3x_i}{2(1 - x_i^2)}(x - x_i)^2\right\} l_i^3(x), \tag{1.9}$$

$$z_i(x) = \frac{1}{2}(x - x_i)^2 l_i^3(x), \tag{1.10}$$

and

$$l_i(x) = \frac{T_n(x)}{T_n'(x_i)(x - x_i)}. \tag{1.11}$$

The interpolation process of the form given by (1.7)–(1.11) was not investigated before. In this paper we prove the convergence of the interpolation process to the function together with the derivatives up to order q. Also we give an estimate for the error. The convergence is given in the following

Theorem 1.1. *Let f be an arbitrary real function defined on $[-1, 1]$. Suppose that $f \in C^{(2q)}([-1, 1])$. Then the following inequalities hold true for all $x \in [-1, 1]$,*

$$|f^{(s)}(x) - H_{3n-1}^{(s)}(x, Y, A)| = O(1)\omega\left(\frac{1}{n}, f^{(2q)}\right)\frac{\log n}{n^{2q-2s}}, \quad s = 1, 2, \ldots, q \tag{1.12}$$

i.e

$$H_{3n-1}^{(q)}(x, Y, A) \to f^{(q)}(x)(n \to \infty) \quad \textit{uniformly if} \quad \omega\left(\frac{1}{n}, f^{(q)}\right) = o(1)[\log n]^{-1}.$$

If $f \in C^{(q)}([-1, 1])$, then

$$|f^{(q)}(x) - H_{3n-1}^{(q)}(x, Y, A)| = O(1)\frac{\omega\left(\frac{1}{n}, f^{(q)}\right)}{(1 - x^2)^{q/2}}\log n \tag{1.13}$$

for $|x| < 1$.

2. Preliminaries. In this section we briefly introduce the most important formulas and definitions needed for our proofs. It is obvious from (1.3) , and (1.10) that

$$T_n'(x_i) = (-1)^n \frac{n}{\sqrt{1-x_i^2}}, \quad 1 \leq i \leq n;\ n = 1, 2, 3, \dots \tag{2.1}$$

and

$$\sum_{i=1}^{n} |l_i^2(x)| \leq 2 \tag{2.2}$$

for $|x| \leq 1$, i.e,

$$|l_i(x)| \leq \sqrt{2}, \qquad i = 1, 2, 3, \dots n,\ |x| \leq 1. \tag{2.3}$$

For the Lebesgue function [5] we have

$$\frac{\log n}{8\sqrt{\pi}} \leq \max_{|x|\leq 1} \sum_{i=1}^{n} |l_i(x)| \leq \frac{2}{\pi} \log n. \tag{2.4}$$

We shall use the well known S. Bernstein's [1] and Markov's [5] inequalities which are given as follows (see respectively [2, 5]).

For any polynomial $g_k(x)$ with real coefficients and degree k, we have

$$|g_k^{(q)}(x)| = O(1) \frac{k^q}{(1-x^2)^{q/2}} \max_{|x|\leq 1} |g_k(x)|, \tag{2.5}$$

$$|g_k^{(q)}(x)| = O(1) k^{2q} \max_{|x|\leq 1} |g_k(x)|. \tag{2.6}$$

For our proofs, we need the following

Theorem 2.1 (I. E. Gopengous [3])**.** *Let $f(x) \in C^{(q)}([-1, 1])$ be a real valued function. Then there exists a polynomial $G_m(x, f)$ of degree at most m ($m \geq 4q + 5$), such that the inequality*

$$|f^{(i)}(x) - G_m^{(i)}(x, f)| = O(1)\omega\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-i}, \tag{2.7}$$

holds true for $i = 0, \ldots, q$ *and for all* $x \in [-1, 1]$. *Here* $\omega(\delta, f^{(q)})$ *is the modulus of continuity of* $f^{(q)}(x)$.

3. Proof of the Theorem 1.1. Let $G(x)$ be the Gopengaus polynomial of degree at most $3n-1$. From theory of interpolation [5] we have

$$G(x) \equiv H_{3n-1}(x, Y_g, A) \tag{3.1}$$

where

$$Y_g = \{G^{(s)}(x_i)\}_{i=1}^{n} \quad (s = 0, 1, 2) \tag{3.2}$$

Using Markov's inequality (2.6) we obtain

$$|G^{(s)}(x) - H_{3n-1}^{(s)}(x, Y, A)| = O(1)n^{2s} \max_{|x|\le 1} |G(x) - H_{3n-1}(x, Y, A)|, \tag{3.3}$$

for $s = 0, 1, 2, \ldots, q$.

Hence from (1.2), (1.3), (1.6)–(1.11), (2.7) together with (3.1) we get for $f \in C^{(2q)}([-1, 1])$

$$|G(x) - H_{3n-1}(x, y, A)| \frac{O(1)\omega\left(\dfrac{1}{3n-1}; f^{(2q)}\right)}{n^{2q}} \{J_1 + J_2 + J_3\}, \tag{3.4}$$

where

$$J_1 = \sum_{i=1}^{n} (1 - x_i^2)^q \left\{|l_i^3(x)| + \frac{|l_i(x)|}{n^2(1 - x_i^2)} + |l_i(x)|\right\},$$

$$J_2 = \sum_{i=1}^{n} (1 - x_i^2)^q \left\{|l_i^2(x)| + \frac{|l_i(x)|}{n\sqrt{1 - x_i^2}}\right\}, \tag{3.5}$$

and

$$J_3 = \sum_{i=1}^{n} (1 - x_i^2)^q |l_i(x)|.$$

Using (2.2)–(2.4) and (1.2) into (3.3) we get

$$\begin{aligned} J_1 &= O(1)\log n \\ J_2 &= O(1) \\ J_3 &= O(1)\log n \end{aligned} \tag{3.6}$$

Thus (3.6) when substituted in (3.4) and the result into (3.3) we get

$$|G^{(s)}(x) - H^{(s)}_{3n-1}(x, Y, A)| = O(1)\omega\left(\frac{1}{n}, f^{(2q)}\right)\frac{\log n}{n^{2q-2s}} \tag{3.7}$$

for all $n \geq \left[\frac{8q}{3}\right] + 2$ and $s = 0, 1, 2, \dots, q$.

Using the triangular inequality, (2.7) and (3.7) we come to

$$\begin{aligned} &|f^{(s)}(x) - H^{(s)}_{3n-1}(x, Y, A)| \leq \\ &\qquad \leq |f^{(s)}(x) - G^{(s)}(x)| + |G^{(s)}(x) - H^{(s)}_{3n-1}(x, Y, A)| = \\ &\qquad = O(1)\omega\left(\frac{1}{n}, f^{(2q)}\right)\frac{\log n}{n^{2q-2s}} \end{aligned} \tag{3.8}$$

for all $n \geq \left[\frac{8q}{3}\right] + 2$ and $s = 0, 1, 2, \dots, q$.

Using S. Bernstein's inequality (2.5) and (3.1) we get

$$\begin{aligned} &|G^{(q)}(x) - H^{(q)}_{3n-1}(x, Y, A)| = \\ &\qquad = O(1)\frac{n^q}{(1 - x^2)^{q/2}} \max_{|x|\leq 1} |G(x) - H_{3n-1}(x, Y, A)|. \end{aligned} \tag{3.9}$$

If $f \in C^{(q)}([-1, 1])$, then from (1.6)–(1.9), (2.7) and (3.1) we obtain for all $n \geq \left[\frac{4q}{3}\right] + 2$,

$$|G(x) - H_{3n-1}(x, A, Y)| = O(1)\frac{\omega\left(\frac{1}{n}, f^{(q)}\right)}{n^q}\{\overline{J}_1 + \overline{J}_2 + \overline{J}_3\} \tag{3.10}$$

where

$$\overline{J}_1 = \sum_{i=1}^{n}(1-x_i^2)^{q/2}\left\{|l_i^3(x)| + \frac{1}{n}\frac{|l_i(x)|}{\sqrt{1-x_i^2}} + |l_i(x)|\right\}, \tag{3.11}$$

$$\overline{J}_2 = \overline{\sum_{i=1}^{n}(1-x_i^2)^{q/2}\left\{l_i^2(x) + \frac{|l_i(x)|}{n\sqrt{1-x_i^2}}\right\}} \tag{3.12}$$

and

$$\overline{J}_3 = \overline{\sum_{i=1}^{n}(1-x_i^2)^{q/2}|l_i(x)|}. \tag{3.13}$$

It is obvious from (2.1)–(2.4) and (3.11)–(3.13) that

$$\overline{\overline{J}}_1 + \overline{J}_2 + \overline{J}_3 = O(1)\log n \tag{3.14}$$

Thus from the triangular inequality, together with (3.9)–(3.10), (3.14) and (2.7) one can easily obtain

$$|f^{(q)}(x) - H^{(q)}_{3n-1}(x,Y,A)|| = O(1)\frac{\omega\left(\dfrac{1}{n}, f^{(q)}\right)}{(1-x^2)^{q/2}}\log n.$$

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